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Unit Groups of Group Algebras of Certain Dihedral Groups

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ABSTRACT

In this article, we give the complete characterization of $U(FD_4)$, $U(FD_8)$, $U(FD_{10})$, $U(FD_{16})$ and $U(FD_{20})$, where F is a finite field of characteristic $p > 0$ and D_n is the dihedral group of order $2n$. We also find the structure of $U(FD_{2^k})$ and $U(FD_{5.2^k})$, when F is a finite field of characteristic 2.

Keywords: Dihedral group, group algebra, unit group.

1. Introduction

Let FG denote the group algebra of a group G over a field F and let $U(FG)$ be the unit group of FG. If H is a normal subgroup of G, then the natural homomorphism $G \to G/H$ can be extended to an F-algebra homomorphism $FG \to F(G/H)$. The kernel of this homomorphism $\omega(H)$, is the ideal of FG generated by $\{h-1 : h \in H\}$. The ideal $\omega(G)$ is called the augmentation ideal of FG and is also denoted by $\omega(FG)$. Clearly, $\omega(H) = \omega(FH)FG = FG\omega(FH)$. We shall be writing $(\omega(H))^n$ as $\omega^n(H)$.

Let $J(FG)$ be the Jacobson radical of FG. For any ideal $I \subseteq J(FG)$, the natural homomorphism $FG \to FG/I$ induces an epimorphism $U(FG) \to U(FG/I)$, so that $U(FG)/(1+I) \cong U(FG/I)$.

Let F be a finite field of characteristic p and let G be a finite group. An element $g \in G$ is called p-regular if $(p, o(g)) = 1$. Let m be the lcm of the orders of p-regular elements of G and let η be the primitive mth root of unity over F. Let T be the multiplicative group of integers t modulo m such that $\eta \to \eta^t$ is an F-automorphism of $F(\eta)$. Two p-regular elements $x, y \in G$ are F-conjugate if $y^t = g^{-1}xg$ for some $g \in G$ and $t \in T$. This is an equivalence relation and partitions the p-regular elements of G into F-conjugacy classes. According to Witt-Berman Theorem (Karpilovsky, 1992, Ch. 17, Theorem 5.3), the number of F-conjugacy classes of p-regular elements of G is equal to the number of non-isomorphic simple FG-modules.

Our notations are standard. For $x, y \in G$, $(x, y) = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$. For a finite subgroup H of G, $\hat{H} = \sum_{h \in H} h$ and $\gamma_n(G) = nth$ term of the lower central series of G. We shall denote by D_n the dihedral group of order $2n$. Thus $D_n = \langle r, s | r^n, s^2, rsrs \rangle$. Also $M(n, F)$ is the algebra of all $n \times n$ matrices over F and $GL(n, F)$ is the general linear group of degree n over F. Further, F_n is the extension field of F of degree n, $F^* = F \backslash \{0\}$ and F^n is the direct summand of n copies of F. C_n is the cyclic group of order n and C_n^k is the direct product of k copies of C_n. The group $K_4 = C_2 \times C_2$.

The study of the unit group of a group ring has been a classical topic in the theory of group rings. Unit groups of several finite group algebras have been described in Creedon (2008), Creedon and Gildea (2008, 2011), Gaohua and Yanyan (2011), Gildea (2008, 2010a,b), Khan (2009), Makhijani et al. (2014a,c, 2015). $U(F_{2^k}D_n)$ has been determined in Makhijani et al. (2014c) for odd n and $U(F_{2^k}D_4)$ and $U(F_{5^k}D_5)$ in terms of split extension have been obtained by Gildea in Creedon and Gildea (2011) and Gildea (2010b) respectively. Complete characterization of $U(FD_6)$ and $U(FD_5)$ has been determined in Gaohua and Yanyan (2011) and Khan (2009) respectively. The conjugacy classes in D_n are as follows:

For odd n:

1. The identity element: $\{1\}$;
2. $(n-1)/2$ conjugacy classes of size 2: $\{r^{\pm 1}\}, \dots, \{r^{\pm(n-1)/2}\}$;
3. All the reflections: $\{r^i s : 0 \leq i \leq (n-1)\}$.

For even n:

1. Two conjugacy classes of size 1: $\{1\}, \{r^{n/2}\}$;
2. $n/2 - 1$ conjugacy classes of size 2: $\{r^{\pm 1}\}, \dots, \{r^{\pm(n/2-1)}\}$;
3. The reflections fall into two conjugacy classes: $\{r^{2i} s : 0 \leq i \leq n/2 - 1\}$ and $\{r^{2i+1} s : 0 \leq i \leq n/2 - 1\}$.

So the number of conjugacy classes in D_n is $(n+6)/2$, if n is even and $(n+3)/2$, if n is odd.

Lemma 1.1. *Creedon (2008)If FG is a semisimple group algebra of an abelian group G over a field F and F contains a primitive mth root of unity, where $m = exp(G)$ and $n = |G|$, then $FG \cong F^n$.*

Lemma 1.2. *Milies and Sehgal (2002)Let RG be a semisimple group algebra. If G' denotes the commutator subgroup of G, then $RG = RG_{e_{G'}} \oplus \Delta(G, G')$, where $RG_{e_{g'}} \cong R(G/G')$ is the sum of all commutative simple components of RG and $\Delta(G, G')$ is the sum of all the others.*

Lemma 1.3. *Milies and Sehgal (2002)(Wedderburn-Artin)A ring R is semisimple if and only if it is a direct sum of finite number of matrix algebras over division rings.*

Lemma 1.4. *Milies and Sehgal (2002) Let G be a group and let R be a commutative ring. The set $\{\gamma_i\}_{i \in I}$ of all class sums is a basis of $Z(RG)$, the center of RG over R.*

The paper is organized as follows. In Section 2, we give a characterization of $U(FD_4)$. Also, we give a description of $U(FD_8)$ and $U(FD_{16})$, if F has odd characteristic. In Section 3, we study $U(FD_{10})$, where F is an arbitrary finite field. We also study $U(FD_{20})$, if F has odd characteristic. Finally, in Section 4, we have investigated $U(FD_{2^k})$ and $U(FD_{5.2^k})$, if F is a field of characteristic 2.

2. Dihedral groups of orders 8, 16 and 32

In this section, we study the structure of $U(FD_{2^n})$ for $n = 2, 3$ and 4.

Theorem 2.1. *Let F be a finite field of characteristic p with $|F| = p^n = q$. Let $V_1 = 1 + J(FD_4)$ and $V_2 = 1 + \omega(D_4')$.*

1. *If $p = 2$, then*
   - *(a) $U(FD_4)/V_1 \cong F^*$;*
   - *(b) V_1 is a group of order 2^{7n};*
   - *(c) V_2 is an abelian group of order 2^{4n};*
   - *(d) V_1/V_2 is a group of exponent 4 and order 2^{3n};*
   - *(e) $U(FD_4)$ is a nilpotent group of class 2.*
2. *If $p > 2$, then $U(FD_4) \cong GL(2, F) \times (F^*)^4$.*

Proof. 1. (a) Let $p = 2$. The commutator subgroup of D_4 is $D_4' = \{1, r^2\}$ and $D_4/D_4' \cong K_4$. Thus $F(D_4/D_4') \cong FD_4/\omega(D_4') \cong FK_4$ and $\dim_F(\omega(D_4')) = 4$. Since $\omega(D_4')$ is nilpotent, by (Lam, 1991, Lemma 4.11), $\omega(D_4') \subseteq J(FD_4)$. Now, $J(FK_4) \cong J(FD_4)/\omega(D_4')$. Let $K_4 = \{1, a, b, ab\}$. Then $J(FK_4) = \alpha_1(1 + ab) + \alpha_2(1 + b) + \alpha_3(1 + a)$; $\alpha_1, \alpha_2, \alpha_3 \in F$, $\dim_F(J(FK_4)) = 3$ and $J^4(FK_4) = 0$. Thus, $\dim_F(J(FD_4)) = 7$, $J^4(FD_4) \subseteq \omega(D_4')$ and $\dim_F(FD_4/J(FD_4)) = 1$. Hence, $FD_4/J(FD_4) \cong F$ and

$$U(FD_4)/V_1 \cong U(FD_4/J(FD_4)) \cong F^*.$$

(b) Since $V_1 = 1 + J(FD_4)$ and $\dim_F(J(FD_4)) = 7$, $|V_1| = |J(FD_4)| = 2^{7n}$.

(c) Since $V_2 = 1 + \omega(D_4')$ and $\omega^2(D_4') = 0$, $V_2' = 1$. Also, $|V_2| = |\omega(D_4')| = 2^{4n}$.

(d) Let $v = v_1V_2 \in V_1/V_2$ where $v_1 = 1 + x \in V_1$, where $x \in J(FD_4)$. As $J^4(FD_4) \subseteq \omega(D_4')$, so $v_1^4 = 1 + x^4 \in V_2$. Hence V_1/V_2 is a group of exponent 4. Further, $|V_1/V_2| = 2^{3n}$.

(e) Since $\omega(D_4')$ is nilpotent, $1+\omega(D_4') \subseteq U(FD_4)$ and $U(FD_4)/(1+\omega(D_4')) \cong U(FK_4)$ is an abelian group. So $U(FD_4)' \subseteq 1+\omega(D_4')$. Further, $\omega(D_4') \subseteq Z(FD_4)$, thus $\gamma_3(U(FD_4)) = 1$ and $U(FD_4)$ is nilpotent of class 2.

2. Since $p > 2$, FD_4 is semisimple. Hence, by Lemma 1.3, $FD_4 \cong M(n_1, D_1) \oplus M(n_2, D_2) \oplus \cdots \oplus M(n_t, D_t)$, where D_i's are finite dimensional division algebras over F. Since F is finite, D_i's are finite fields and at least one $n_k > 1$. As $\dim_F Z(FD_4) = 5$, $n_k > 2$ is impossible. Therefore $n_k \leq 2$ for all $k \in \{1, 2, \ldots, t\}$. Also $F(D_4/D_4') \cong FK_4 \cong F^4$ and $FD_4 \cong M(2, F) \oplus F^4$, by Lemmas 1.1 and 1.2.

□

Theorem 2.2. *Let F be a finite field of characteristic $p > 2$ with $|F| = p^n = q$. Then*

$$U(FD_8) \cong \begin{cases} GL(2,F)^3 \times C_{q-1}^4, & \text{if } q \equiv \pm 1 \text{ mod } 8 \text{ ;} \\ GL(2,F) \times GL(2,F_2) \times C_{q-1}^4, & \text{if } q \equiv \pm 3 \text{ mod } 8. \end{cases}$$

Proof. Since $p > 2$, FD_8 is semisimple and by Lemma 1.3, $FD_8 \cong M(n_1, D_1) \oplus M(n_2, D_2) \oplus \cdots \oplus M(n_t, D_t)$, where D_i's are finite dimensional division algebras over F. Since F is finite, D_i's are finite fields and at least one $n_k > 1$. Clearly $n_k \leq 3$ for all k. Now $F(D_8/D_8') \cong FK_4 \cong F^4$. Further, since $\dim_F Z(FD_8) = 7$, we have the following possibilities:

$$FD_8 \cong M(2,F) \oplus M(2,F) \oplus M(2,F) \oplus F^4$$

or

$$\cong M(2,F) \oplus M(2,F_2) \oplus F^4$$

or

$$\cong M(2,F_3) \oplus F^4.$$

The conjugacy classes of D_8 are $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r^4\}$, $\mathcal{C}_3 = \{r, r^7\}$, $\mathcal{C}_4 = \{r^2, r^6\}$, $\mathcal{C}_5 = \{r^3, r^5\}$, $\mathcal{C}_6 = \{rs, r^3s, r^5s, r^7s\}$ and $\mathcal{C}_7 = \{s, r^2s, r^4s, r^6s\}$ and by Lemma 1.4, $Z(FD_8) = F\widehat{\mathcal{C}_1} + F\widehat{\mathcal{C}_2} + F\widehat{\mathcal{C}_3} + F\widehat{\mathcal{C}_4} + F\widehat{\mathcal{C}_5} + F\widehat{\mathcal{C}_6} + F\widehat{\mathcal{C}_7}$.

If $p \equiv \pm 1$ mod 8, then $p^n \equiv \pm 1$ mod 8 for all n. So, $\widehat{\mathcal{C}_i}^{p^n} = \widehat{\mathcal{C}_i}$ for all, $1 \leq i \leq 7$. Thus $x^{p^n} = x$, for all $x \in Z(FD_8)$ and

$$FD_8 \cong M(2,F) \oplus M(2,F) \oplus M(2,F) \oplus F^4.$$

If $p \equiv \pm 3 \bmod 8$ and n is even, then $p^n \equiv 1 \bmod 8$. Again, $\widehat{\mathcal{C}_i}^{p^n} = \widehat{\mathcal{C}_i}$ for all, $1 \leq i \leq 7$ and

$$FD_8 \cong M(2,F) \oplus M(2,F) \oplus\ M(2,F) \oplus F^4.$$

If $p \equiv \pm 3 \bmod 8$ and n is odd, then $p^{2n} \equiv 1 \bmod 8$. So, $\widehat{\mathcal{C}_i}^{p^{2n}} = \widehat{\mathcal{C}_i}$ for all $1 \leq i \leq 7$. Then, in this case, $x^{p^{2n}} = x$, for any $x \in Z(FD_8)$ and

$$FD_8 \cong M(2,F) \oplus\ M(2,F_2) \oplus F^4.$$

Hence

$$FD_8 \cong \begin{cases} M(2,F)^3 \oplus F^4, & \text{if } q \equiv \pm 1 \bmod 8\ ; \\ M(2,F) \oplus M(2,F_2) \oplus F^4, & \text{if } q \equiv \pm 3 \bmod 8\ . \end{cases}$$

□

Theorem 2.3. *Let F be a finite field of characteristic $p > 2$ with $|F| = p^n = q$. Then*

$$U(FD_{16}) \cong \begin{cases} GL(2,F)^7 \times C_{q-1}^4, & \text{if } q \equiv \pm 1 \bmod 16; \\ GL(2,F) \times GL(2,F_2) \times GL(2,F_4) \times C_{q-1}^4, & \text{if } q \equiv \pm 3 \text{ or } \pm 5 \bmod 16; \\ GL(2,F)^3 \times GL(2,F_2)^2 \times C_{q-1}^4, & \text{if } q \equiv \pm 7 \bmod 16. \end{cases}$$

Proof. As, $F(D_{16}/D'_{16}) \cong FK_4 \cong F^4$, we have

$$FD_{16} \cong F^4 \oplus \Big(\oplus_{i=1}^k M(n_i, D_i) \Big)$$

where $n_i \geq 2$ and D_i's are finite dimensional division algebras over fields which are extensions of F. Hence,

$$Z(FD_{16}) \cong F^4 \oplus \Big(\oplus_{i=1}^k D_i \Big).$$

Since $\dim_F Z(FD_{16}) = 11$, $\sum_{i=1}^k [D_i : F] = 7$.

The conjugacy classes of D_{16} are $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r^8\}$, $\mathcal{C}_3 = \{r^{\pm 1}\}$, $\mathcal{C}_4 = \{r^{\pm 2}\}$, $\mathcal{C}_5 = \{r^{\pm 3}\}$, $\mathcal{C}_6 = \{r^{\pm 4}\}$, $\mathcal{C}_7 = \{r^{\pm 5}\}$, $\mathcal{C}_8 = \{r^{\pm 6}\}$, $\mathcal{C}_9 = \{r^{\pm 7}\}$, $\mathcal{C}_{10} = \{s, r^2 s, \ldots, r^{14} s\}$ and $\mathcal{C}_{11} = \{rs, r^3 s, \ldots, r^{15} s\}$.

For any $l \in \mathbb{N}$, it is easy to see that $x^{q^l} = x$ for all $x \in Z(FD_{16})$ if and only if $\widehat{\mathcal{C}}_i^{q^l} = \widehat{\mathcal{C}_i}$ for all $i \in \{1, 2, \ldots, 11\}$. This is possible if and only if $r^{q^l} = r$ or r^{-1}. This is equivalent to $16|(q^l - 1)$ or $16|(q^l + 1)$.

Now for each $i \in \{1, 2, \ldots, k\}$, let $D_i^* = \langle y_i \rangle$. Then, $x^{q^l} = x$ for all $x \in Z(FD_{16})$ if and only if $y_i^{q^l} = y_i$. This is possible if and only if $[D_i : F] | l$ for all $i = 1, \ldots, k$. Thus the least number t such that $16 | (q^t - 1)$ or $16 | (q^t + 1)$ is $t = l.c.m. \{[D_i : F] : 1 \leq i \leq k\}$. Now if,

1. $q \equiv \pm 1 \bmod 16$, then $t = 1$;
2. $q \equiv \pm 3$ or $\pm 5 \bmod 16$, then $t = 4$;
3. $q \equiv \pm 7 \bmod 16$, then $t = 2$.

Clearly $m = 16$. Let $a=$ number of simple components in the Wedderburn decomposition of FD_{16}. Then

1. $q \equiv 1 \bmod 16$.

   $T = \{1\} \bmod 16$ and hence $\mathcal{C}_i$, $i \in \{1, 2, \ldots, 11\}$ are the p-regular F-conjugacy classes. Hence $a = 11$.

2. $q \equiv -1 \bmod 16$.

   $T = \{1, -1\} \bmod 16$ and hence $\mathcal{C}_i$, $i \in \{1, 2, \ldots, 11\}$ are the p-regular F-conjugacy classes. Hence $a = 11$.

3. $q \equiv 3$ or $-5 \bmod 16$.

   $T = \{1, 3, 9, 11\} \bmod 16$. Since $r^9 = r^{-7}$, $r^{11} = r^{-5}$ and $(r^2)^3 = r^6$, the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 3}, r^{\pm 5}, r^{\pm 7}\}$, $\{r^{\pm 2}, r^{\pm 6}\}$, $\{r^{\pm 4}\}$, $\{r^8\}$, $\{rs, r^3 s, \ldots, r^{15} s\}$ and $\{s, r^2 s, \ldots, r^{14} s\}$. Hence $a = 7$.

4. $q \equiv 5$ or $-3 \bmod 16$.

   $T = \{1, 5, 9, 13\} \bmod 16$. Since $r^9 = r^{-7}$, $r^{13} = r^{-3}$ also $(r^2)^5 = r^{-6}$, the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 3}, r^{\pm 5}, r^{\pm 7}\}$, $\{r^{\pm 2}, r^{\pm 6}\}$, $\{r^{\pm 4}\}$, $\{r^8\}$, $\{rs, r^3 s, \ldots, r^{15} s\}$ and $\{s, r^2 s, \ldots, r^{14} s\}$. Hence $a = 7$.

5. $q \equiv 7 \bmod 16$

   $T = \{1, 7\} \bmod 16$. Since $(r^3)^7 = r^5$, the p-regular F-conjugacy classes are given by $\{1\}$, $\{r^{\pm 1}, r^{\pm 7}\}$, $\{r^{\pm 2}\}$, $\{r^{\pm 3}, r^{\pm 5}\}$, $\{r^{\pm 4}\}$, $\{r^{\pm 6}\}$, $\{r^8\}$, $\{rs, r^3 s, \ldots, r^{15} s\}$ and $\{s, r^2 s, \ldots, r^{14} s\}$. Hence $a = 9$.

6. $q \equiv -7 \bmod 16$.

   $T = \{1, 9\} \bmod 16$. Since $r^9 = r^{-7}$, $(r^3)^9 = r^{-5}$, the p-regular F-conjugacy classes are $\{1\}, \{r^{\pm 1}, r^{\pm 7}\}$, $\{r^{\pm 2}\}$, $\{r^{\pm 3}, r^{\pm 5}\}$, $\{r^{\pm 4}\}$, $\{r^{\pm 6}\}$, $\{r^8\}$, $\{rs, r^3 s, \ldots, r^{15} s\}$ and $\{s, r^2 s, \ldots, r^{14} s\}$. Hence $a = 9$.

Now, we have the following possibilities for $[D_i : F]_{i=1}^k$ depending on q:

1. $q \equiv \pm 1 \bmod 16$, then $[D_i : F]_{i=1}^k = (1, 1, 1, 1, 1, 1, 1)$.
2. $q \equiv \pm 3$ or $\pm 5 \bmod 16$, then $[D_i : F]_{i=1}^k = (1, 2, 4)$.
3. $q \equiv \pm 7 \bmod 16$, then $[D_i : F]_{i=1}^k = (1, 1, 1, 2, 2)$.

Due to dimension constraints, $n_i = 2$ for all $1 \leq i \leq k$. Hence

$$FD_{16} \cong \begin{cases} M(2,F)^7 \oplus F^4, & \text{if } q \equiv \pm 1 \bmod 16; \\ M(2,F) \oplus M(2,F_2) \oplus M(2,F_4) \oplus F^4, & \text{if } q \equiv \pm 3 \text{ or } \pm 5 \bmod 16; \\ M(2,F)^3 \oplus M(2,F_2)^2 \oplus F^4, & \text{if } q \equiv \pm 7 \bmod 16. \end{cases}$$

□

3. Dihedral groups of orders 20 and 40

In this section, we find the structure of $U(FD_{5n})$, $n = 2, 4$.

Theorem 3.1. *Let F be a finite field of characteristic p with $|F| = q = p^n$. Let $V_1 = 1 + J(FD_{10})$ and let $V_2 = 1 + \omega(H)$, where $H = \{1, r^5\}$. Then*

1. *If $p = 2$, then*
   (a)
   $$U(FD_{10})/V_1 \cong \begin{cases} GL(2,F)^2 \times C_{2^n-1}, & \text{if } n \text{ is odd;} \\ GL(2,F_2) \times C_{2^n-1}, & \text{if } n \text{ is even.} \end{cases}$$
   (b) *V_1 is a group of exponent 4, order 2^{11n} and nilpotent of class at most 3;*
   (c) *V_2 is an abelian group of order 2^{10n};*
   (d) *V_1/V_2 is a group of exponent 2 and order 2^n.*
2. *If $p = 5$, then*
   (a) *$U(FD_{10})/V_1 \cong C_{5^n-1}^4$;*
   (b) *V_1 is a nilpotent group of class 4 and order 5^{16n}.*

3. If $p \nmid |D_{10}|$, then

$$U(FD_{10}) \cong \begin{cases} GL(2,F)^4 \times C_{q-1}^4, & \text{if } q \equiv \pm 1 \text{ mod } 10; \\ GL(2,F_2)^2 \times C_{q-1}^4, & \text{if } q \equiv \pm 3 \text{ mod } 10. \end{cases}$$

Proof. 1. (a) Let $p = 2$. Then $H = \{1, r^5\}$ is a normal subgroup of D_{10} and $D_{10}/H \cong D_5$. Thus $F(D_{10}/H) \cong FD_{10}/\omega(H) \cong FD_5$ and $\dim_F(\omega(H)) = 10$. Since $\omega(H)$ is a nilpotent ideal, $\omega(H) \subseteq J(FD_{10})$. Now, $J(FD_5) \cong J(FD_{10})/\omega(H)$. By (Makhijani et al., 2014b, Theorem 3.1), $J(FD_5) = F\widehat{D_5}$ and $\dim_F(J(FD_5)) = 1$. So, $\dim_F(J(FD_{10})) = 11$ and $\dim_F(FD_{10}/J(FD_{10})) = 9$.

Now, the 2-regular elements in D_{10} are 1, r^2, r^{-2}, r^4 and r^{-4}. Hence $m = 5$. Let a be the number of simple components in the Wedderburn decomposition of FD_{10}.

i. If $n = 0$ mod 4, then $q \equiv 1$ mod 5.
$T = \{1\}$ mod 5 and $\{1\}, \{r^{\pm 2}\}$, $\{r^{\pm 4}\}$ are the 2-regular F-conjugacy classes. Hence $a = 3$.

ii. If $n = 2$ mod 4, then $q \equiv -1$ mod 5.
$T = \{1, 4\}$ mod 5 and $\{1\}, \{r^{\pm 2}\}$, $\{r^{\pm 4}\}$ are the 2-regular F-conjugacy classes. Hence $a = 3$.

iii. If $n = 1$ mod 2, then $q \equiv \pm 2$ mod 5.
$T = \{1, 2, 3, 4\}$ mod 5 and $\{1\}, \{r^{\pm 2}, r^{\pm 4}\}$ are the 2-regular F-conjugacy classes. Hence $a = 2$.

Hence,

$$FD_{10}/J(FD_{10}) \cong \begin{cases} F \oplus M(2,F)^2, & \text{if } q \equiv \pm 1 \text{ mod } 5; \\ F \oplus M(2,F_2), & \text{if } q \equiv \pm 2 \text{ mod } 5. \end{cases}$$

(b) Since $J^2(FD_5) = 0$, so $J^2(FD_{10}) \subseteq \omega(H)$ and $J^4(FD_{10}) = 0$. Hence V_1 is a group of exponent 4 which is nilpotent of class at most 3. Further, since $V_1 = 1 + J(FD_{10})$, $|V_1| = |J(FD_{10})| = 2^{11n}$.

(c) Since $V_2 = 1 + \omega(H)$ and $\omega^2(H) = 0$, $V_2' = 1$. Hence V_2 is abelian. Further, $|V_2| = |\omega(H)| = 2^{10n}$.

(d) Let $v = v_1 V_2 \in V_1/V_2$ where $v_1 \in V_1$. For $x \in J(FD_{10})$, let $v_1 = 1 + x$ so that $v_1^2 = 1 + x^2 \in V_2$. Hence V_1/V_2 is a group of exponent 2. Further, $|V_1/V_2| = 2^n$.

2. (a) Let $p = 5$ and let $K = \{1, r^{\pm 2}, r^{\pm 4}\}$. Then K is a normal subgroup of D_{10}. By (Passman, 1977, Lemma 1.17 and Theorem 2.7),

$J(FD_{10}) = \omega(K)$. Thus $FD_{10}/J(FD_{10}) \cong FK_4 \cong F^4$ by Lemma 1.1 and $\dim_F(J(FD_{10})) = 16$. Hence,

$$U(FD_{10})/V_1 \cong U(FD_{10}/J(FD_{10})) \cong (F^*)^4.$$

(b) $J^5(FD_{10}) = \omega^5(K) = 0$. Hence V_1 is nilpotent of class at most 4. As $r^2+4 \in J(FD_{10})$, so $4s+r^8s$, $4r+r^3$, $4rs+r^9s \in J(FD_{10})$. Thus $x = r^2$, $y = 1+4s+r^8s$, $z = 1+4r+r^3$ and $w = 1+4rs+r^9s \in V_1$. Now,

$$\begin{aligned}
A &= (x,y) = 3 + 4r^2 + r^4 + 3r^6 + (1 + 2r^2 + r^4 + 3r^6 + 3r^8)s,\\
B &= (z,A) = 2 + r^3 + r^5 + r^6 + 3r^8 + r^9 + (3 + 4r + 4r^2 + 2r^3\\
&\quad + 2r^5 + 4r^6 + r^7 + r^8 + 4r^9)s,\\
C &= (w,B) = 4 + 3r + 2r^2 + 3r^3 + r^4 + 3r^6 + 3r^8 + 2r^9\\
&\quad + (4r + 2r^2 + 4r^5 + 3r^6 + r^7 + 3r^8 + r^9)s \neq 1.
\end{aligned}$$

Hence V_1 is nilpotent of class 4. In the above expression,

$$\begin{aligned}
y^{-1} &= 4 + r^4 + r^6 + (4 + 4r^2 + r^6 + r^8)s,\\
A^{-1} &= 3 + r^6 + 3r^4 + 4r^8 + (4 + 3r^2 + 4r^4 + 2r^6 + 2r^8)s,\\
z^{-1} &= 2 + r + 2r^2 + 4r^4 + 2r^5 + 2r^6 + 3r^7 + r^8 + 4r^9,\\
B^{-1} &= 2 + 2r + 3r^2 + 2r^3 + r^6 + 2r^7 + 4r^8\\
&\quad + 2r^9 + (1 + 4r + r^3 + r^4 + 4r^5 + 3r^6 + 4r^7 + 2r^8 + 2r^9)s,\\
w^{-1} &= 2 + 4r^6 + (3r + 2r^7)s.
\end{aligned}$$

Further, since $V_1 = 1 + J(FD_{10})$, $|V_1| = |J(FD_{10})| = 5^{16n}$.

3. As $F(D_{10}/D'_{10}) \cong FK_4 \cong F^4$, so by using the Wedderburn-Artin Theorem and Lemma 1.2, we have

$$FD_{10} \cong F^4 \oplus \Big(\oplus_{i=1}^{k} M(n_i, D_i)\Big),$$

where $n_i \geq 2$ and D_i's are finite fields. Therefore,

$$Z(FD_{10}) \cong F^4 \oplus \Big(\oplus_{i=1}^{k} D_i\Big).$$

Since, $\dim_F Z(FD_{10}) = 8$, $\sum_{i=1}^{k}[D_i : F] = 4$.
The conjugacy classes of D_{10} are $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r^5\}$, $\mathcal{C}_3 = \{r^{\pm1}\}$, $\mathcal{C}_4 = \{r^{\pm2}\}$, $\mathcal{C}_5 = \{r^{\pm3}\}$, $\mathcal{C}_6 = \{r^{\pm4}\}$, $\mathcal{C}_7 = \{s, r^2s, \ldots, r^8s\}$ and $\mathcal{C}_8 = \{rs, r^3s, \ldots, r^9s\}$.

For any $l \in \mathbb{N}$, it is easy to see that $x^{q^l} = x$ for all $x \in Z(FD_{10})$ if and only if $\widehat{\mathcal{C}}_i^{q^l} = \widehat{\mathcal{C}}_i$ for all $1 \leq i \leq 8$. This is possible, if and only if $r^{q^l} = r$ or r^{-1} or equivalently $10|(q^l - 1)$ or $10|(q^l + 1)$.

Now for each i, $1 \leq i \leq k$, let $D_i^* = \langle y_i \rangle$. Then, $x^{q^l} = x$ for all $x \in Z(FD_{10})$ if and only if $y_i^{q^l} = y_i$. This is possible if and only if $[D_i : F]|l$ for all $i \in \{1, \ldots, k\}$. Thus the least number t such that $10|(q^t - 1)$ or $10|(q^t + 1)$ is $t = l.c.m.\{[D_i : F] : 1 \leq i \leq k\}$.

If

(a) $q \equiv \pm 1 \bmod 10$, then $t = 1$.

(b) $q \equiv \pm 3 \bmod 10$, then $t = 2$.

We have $m = 10$. Let $a=$ number of simple components in the Wedderburn decomposition of FD_{10}. Then

(a) $q \equiv 1 \bmod 10$.
$T = \{1\} \bmod 10$, so $\mathcal{C}_i, 1 \leq i \leq 8$ are the p-regular F-conjugacy classes. Hence $a = 8$.

(b) $q \equiv -1 \bmod 10$.
$T = \{1, -1\} \bmod 10$, so $\mathcal{C}_i, 1 \leq i \leq 8$ are the p-regular F-conjugacy classes. Hence $a = 8$.

(c) $q \equiv \pm 3 \bmod 10$.
$T = \{1, 3, 7, 9\} \bmod 10$. Since $r^7 = r^{-3}$, $r^9 = r^{-1}$ and $(r^2)^3 = r^{-4}$, the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 3}\}$, $\{r^{\pm 2}, r^{\pm 4}\}$, $\{r^5\}$, $\{s, r^2 s, \ldots, r^8 s\}$ and $\{rs, r^3 s, \ldots, r^9 s\}$. Hence $a = 6$.

Now, we have the following possibilities for $[D_i : F]_{i=1}^k$ depending on q,

(a) $q \equiv \pm 1 \bmod 10$, then $[D_i : F]_{i=1}^k = (1, 1, 1, 1)$.

(b) $q \equiv \pm 3 \bmod 10$, then $[D_i : F]_{i=1}^k = (2, 2)$.

Due to dimension constraints, $n_i > 2$ is impossible for any $1 \leq i \leq k$. Thus $n_i = 2$ for all $1 \leq i \leq k$ and

$$FD_{10} \cong \begin{cases} M(2, F)^4 \oplus F^4, & \text{if } q \equiv \pm 1 \bmod 10; \\ M(2, F_2)^2 \oplus F^4, & \text{if } q \equiv \pm 3 \bmod 10. \end{cases}$$

□

Theorem 3.2. *Let F be a finite field of characteristic p with $|F| = q = p^n$. Let $V_1 = 1 + J(FD_{20})$.*

1. *If $p = 5$, then*

   (a) *$U(FD_{20})/V_1 \cong GL(2,F) \times C^4_{5^n-1}$;*

   (b) *V_1 is a nilpotent group of class 4 and order 5^{32n}.*

2. *If $p \nmid |D_{20}|$, then*

$$U(FD_{20}) \cong \begin{cases} GL(2,F)^9 \times C^4_{q-1}, & \text{if } q \equiv \pm 1 \text{ mod } 20; \\ GL(2,F) \times GL(2,F_2)^2 \times GL(2,F_4) \times C^4_{q-1}, & \text{if } q \equiv \pm 3 \text{ or } \pm 7 \text{ mod } 20; \\ GL(2,F)^5 \times GL(2,F_2)^2 \times C^4_{q-1}, & \text{if } q \equiv \pm 9 \text{ mod } 20; \end{cases}$$

Proof. 1. (a) Let $p = 5$ and let $H = \{1, r^{\pm 4}, r^{\pm 8}\}$. Then H is a normal subgroup of D_{20}. Again by (Passman, 1977, Lemma 1.17 and Theorem 2.7), $J(FD_{20}) = \omega(H)$. Thus $FD_{20}/J(FD_{20}) \cong FD_4 \cong M(2,F) \oplus F^4$, by Theorem 2.1 and $\dim_F J(FD_{20}) = 32$. Hence,

$$U(FD_{20})/V_1 \cong U(FD_{20}/J(FD_{20})) \cong GL(2,F) \times (F^*)^4.$$

(b) $J^5(FD_{20}) = \omega^5(H) = 0$. Hence V_1 is nilpotent of class at most 4. As $h = r^4 - 1$ and $k = s(r^4-1) = r^{16}s - s \in J(FD_{20})$, so $x = r^4$ and $y = 1 - s + r^{16}s \in V_1$. Then,

$$\begin{aligned} A &= (x,y) = 3 + 4r^4 + r^8 + 3r^{12} + (1 + 2r^4 + r^8 + 3r^{12} + 3r^{16})s, \\ B &= (x,A) = 1 + (4r^4 + 3r^8 + 2r^{12} + r^{16})s, \\ C &= (x,B) = 1 + (2 + 2r^4 + 2r^8 + 2r^{12} + 2r^{16})s \neq 1. \end{aligned}$$

Hence V_1 is nilpotent of class 4. In the above expression,

$$\begin{aligned} y^{-1} &= 4 + r^8 + r^{12} + (4 + 4r^4 + r^{12} + r^{16})s, \\ A^{-1} &= 3 + 3r^8 + r^{12} + 4r^{16} + (4 + 3r^4 + 4r^8 + 2r^{12} + 2r^{16})s, \\ B^{-1} &= 1 + (r^4 + 2r^8 + 3r^{12} + 4r^{16})s. \end{aligned}$$

Further, since $V_1 = 1 + J(FD_{20})$, $|V_1| = |J(FD_{20})| = 5^{32n}$.

2. Now $F(D_{20}/D'_{20}) \cong FK_4 \cong F^4$. Hence

$$FD_{20} \cong F^4 \oplus \Big(\oplus_{i=1}^k M(n_i, D_i) \Big),$$

where $n_i \geq 2$ and D_i's are finite fields. Therefore,

$$Z(FD_{20}) \cong F^4 \oplus \Big(\oplus_{i=1}^{k} D_i \Big).$$

Since, $\dim_F Z(FD_{20}) = 13$, $\sum_{i=1}^{k}[D_i : F] = 9$.

The conjugacy classes of D_{20} are given by $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{r^{10}\}$, $\mathcal{C}_3 = \{r^{\pm 1}\}$, $\mathcal{C}_4 = \{r^{\pm 2}\}$, $\mathcal{C}_5 = \{r^{\pm 3}\}$, $\mathcal{C}_6 = \{r^{\pm 4}\}$, $\mathcal{C}_7 = \{r^{\pm 5}\}$, $\mathcal{C}_8 = \{r^{\pm 6}\}$, $\mathcal{C}_9 = \{r^{\pm 7}\}$, $\mathcal{C}_{10} = \{r^{\pm 8}\}$, $\mathcal{C}_{11} = \{r^{\pm 9}\}$, $\mathcal{C}_{12} = \{s, r^2s, \ldots, r^{18}s\}$ and $\mathcal{C}_{13} = \{rs, r^3s, \ldots, r^{19}s\}$.

Now for any $l \in \mathbb{N}$, we have $x^{q^l} = x$ for all $x \in Z(FD_{20})$ if and only if $\widehat{\mathcal{C}}_i^{q^l} = \widehat{\mathcal{C}}_i$ for all $1 \leq i \leq 13$. This is possible if and only if $r^{q^l} = r$ or r^{-1} or equivalently $20|(q^l - 1)$ or $20|(q^l + 1)$.

For each i, $1 \leq i \leq k$, let $D_i^* = \langle y_i \rangle$. Then $x^{q^l} = x$ for all $x \in Z(FD_{20})$ if and only if $y_i^{q^l} = y_i$. This is possible if and only if $[D_i : F]|l$ for all $i \in \{1, \ldots, k\}$. Thus the least number t such that $20|(q^t - 1)$ or $20|(q^t + 1)$ is $t = l.c.m.\{[D_i : F] : 1 \leq i \leq k\}$.

If

(a) $q \equiv \pm 1 \bmod 20$, then $t = 1$.

(b) $q \equiv \pm 3$ or $\pm 7 \bmod 20$, then $t = 4$.

(c) $q \equiv \pm 9 \bmod 20$, then $t = 2$.

Clearly $m = 20$. Let $a=$ number of simple components in the Wedderburn decomposition of FD_{20}. Then

(a) $q \equiv 1 \bmod 20$.
$T = \{1\} \bmod 20$ and hence $\mathcal{C}_i, 1 \leq i \leq 13$ are the p-regular F-conjugacy classes. Hence $a = 13$.

(b) $q \equiv -1 \bmod 20$.
$T = \{1, -1\} \bmod 20$ and hence $\mathcal{C}_i, 1 \leq i \leq 13$ are the p-regular F-conjugacy classes. Hence $a = 13$.

(c) $q \equiv 3$ or $7 \bmod 20$.
$T = \{1, 3, 7, 9\} \bmod 20$. Since $(r^2)^3 = r^6$ and $(r^4)^7 = r^8$, the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 3}, r^{\pm 7}, r^{\pm 9}\}$, $\{r^{\pm 5}\}$, $\{r^{10}\}$, $\{r^{\pm 2}, r^{\pm 6}\}$, $\{r^{\pm 4}, r^{\pm 8}\}$, $\{s, r^2s, \ldots, r^{18}s\}$, $\{rs, r^3s, \ldots, r^{19}s\}$. Hence $a = 8$.

(d) $q \equiv 9 \bmod 20$.
$T = \{1, 9\} \bmod 20$. Since $(r^3)^9 = r^7$, the p-regular F-conjugacy classes are $\{1\}$, $\{r^{\pm 1}, r^{\pm 9}\}$, $\{r^{\pm 3}, r^{\pm 7}\}$, $\{r^{\pm 2}\}$, $\{r^{\pm 4}\}$, $\{r^{\pm 5}\}$, $\{r^{\pm 6}\}$,

$\{r^{\pm 8}\}$, $\{r^{10}\}$, $\{s, r^2s, \ldots, r^{18}s\}$ and $\{rs, r^3s, \ldots, r^{19}s\}$. Hence $a = 11$.

(e) $q \equiv -9 \bmod 20$.
$T = \{1, 11\} \bmod 20$. Since $r^{11} = r^{-9}$ and $(r^3)^{11} = r^{-7}$, the p-regular F-conjugacy classes are given by $\{1\}$, $\{r^{\pm 1}, r^{\pm 9}\}$, $\{r^{\pm 2}\}$, $\{r^{\pm 3}, r^{\pm 7}\}$, $\{r^{\pm 4}\}$, $\{r^{\pm 5}\}$, $\{r^{\pm 6}\}$, $\{r^{\pm 8}\}$, $\{r^{10}\}$, $\{s, r^2s, \ldots, r^{18}s\}$ and $\{rs, r^3s, \ldots, r^{19}s\}$. Hence $a = 11$.

(f) $q \equiv -3$ or $-7 \bmod 20$.
$T = \{1, 9, 13, 17\} \bmod 20$. Since $r^{13} = r^{-7}$, $r^{17} = r^{-3}$, $(r^2)^{17} = r^{-6}$ and $(r^4)^{17} = r^8$, the p- regular F-conjugacy classes are given by $\{1\}$, $\{r^{\pm 1}, r^{\pm 3}, r^{\pm 7}, r^{\pm 9}\}$, $\{r^{\pm 2}, r^{\pm 6}\}$, $\{r^{\pm 4}, r^{\pm 8}\}$, $\{r^{\pm 5}\}$, $\{r^{10}\}$, $\{s, r^2s, \ldots, r^{18}s\}$ and $\{rs, r^3s, \ldots, r^{19}s\}$. Hence $a = 8$.

Now, we have the following possibilities for $[D_i : F]_{i=1}^k$ depending on q,

(a) $q \equiv \pm 1 \bmod 20$, then $[D_i : F]_{i=1}^k = (1, 1, 1, 1, 1, 1, 1, 1, 1)$.
(b) $q \equiv \pm 3$ or $\pm 7 \bmod 20$, then $[D_i : F]_{i=1}^k = (1, 2, 2, 4)$.
(c) $q \equiv \pm 9 \bmod 20$, then $[D_i : F]_{i=1}^k = (1, 1, 1, 1, 1, 2, 2)$.

Due to dimension constraints, $n_i > 2$ is impossible for any $1 \leq i \leq k$. Thus $n_i = 2$ for all $1 \leq i \leq k$ and

$$FD_{20} \cong \begin{cases} M(2,F)^9 \oplus F^4, & \text{if } q \equiv \pm 1 \bmod 20; \\ M(2,F) \oplus M(2,F_2)^2 \oplus M(2,F_4) \oplus F^4, & \text{if } q \equiv \pm 3 \text{ or } \pm 7 \bmod 20; \\ M(2,F)^5 \oplus M(2,F_2)^2 \oplus F^4, & \text{if } q \equiv \pm 9 \bmod 20. \end{cases}$$

□

4. Finite fields of characteristic 2

In this section, we find the structure of $U(FD_{2^k})$ and $U(FD_{5.2^k})$ over finite fields of characteristic 2.

Lemma 4.1. *Let F be a finite field of characteristic 2. Then $\dim_F(J(FD_{2^k})) = 2^{k+1} - 1$ for all $k = 2, 3, \ldots$.*

Proof. If $k = 2$, then by Theorem 2.1, $\dim_F(J(FD_4)) = 7$. Suppose that $\dim_F(J(FD_{2^{k-1}})) = 2^k - 1$. If $H = \langle r^{2^{k-1}} \rangle = \{1, r^{2^{k-1}}\}$, then H is normal in D_{2^k} and $F(D_{2^k}/H) \cong FD_{2^k}/\omega(H) \cong FD_{2^{k-1}}$. So $\dim_F(\omega(H)) = 2^k$. Now $\omega(H)$ is nilpotent and so, $\omega(H) \subseteq J(FD_{2^k})$. Therefore $J(FD_{2^{k-1}}) \cong J(FD_{2^k})/\omega(H)$ and $\dim_F(J(FD_{2^k})) = 2^k - 1 + 2^k = 2^{k+1} - 1$. □

Theorem 4.1. *Let F be a finite field of characteristic 2 with 2^n elements. Let $V_1 = 1 + J(FD_{2^k})$ and $V_2 = 1 + \omega(H)$, where $H = \langle r^{2^{k-1}} \rangle$. Then*

1. $U(FD_{2^k})/V_1 \cong F^*$;
2. V_1/V_2 *is a group of order* $2^{(2^k-1)n}$.

Proof. 1. Let $H = \{1, r^{2^{k-1}}\}$. Then H is normal subgroup of D_{2^k}. So $D_{2^k}/H \cong D_{2^{k-1}}$. Thus $FD_{2^k}/\omega(H) \cong FD_{2^{k-1}}$ and $\dim_F(\omega(H)) = 2^k$. Now $\omega(H)$ is nilpotent and so, $\omega(H) \subseteq J(FD_{2^k})$. Thus $J(FD_{2^{k-1}}) \cong J(FD_{2^k})/\omega(H)$ and by Lemma 4.1, $\dim_F(J(FD_{2^k})) = 2^{k+1} - 1$. So, $\dim_F(FD_{2^k}/J(FD_{2^k})) = 1$, $FD_{2^k}/J(FD_{2^k}) \cong F$ and $U(FD_{2^k})/V_1 \cong U(FD_{2^k}/J(FD_{2^k})) \cong F^*$.

2. Obviously, $|V_1| = |J(FD_{2^k})| = 2^{(2^{k+1}-1)n}$ and $|V_2| = |\omega(H)| = 2^{(2^k)n}$. Hence $|V_1/V_2| = 2^{(2^k-1)n}$.

□

Lemma 4.2. *Let F be a finite field of characteristic 2. Then $\dim_F(J(FD_{5.2^k})) = 5.2^{k+1} - 9$, for all $k = 0, 1, 2, \ldots$.*

Proof. If $k = 0$, then by (Makhijani et al., 2014b, Theorem 3.1), $\dim_F(J(FD_5)) = 1$. Suppose that $\dim_F(J(FD_{5.2^{k-1}})) = 5.2^k - 9$ and let $H = \langle r^{5.2^{k-1}} \rangle = \{1, r^{5.2^{k-1}}\}$. Then H is normal in $D_{5.2^k}$ and hence $F(D_{5.2^k}/H) \cong FD_{5.2^k}/\omega(H) \cong FD_{5.2^{k-1}}$. So $\dim_F(\omega(H)) = 5.2^k$. As $\omega(H)$ is nilpotent, $\omega(H) \subseteq J(FD_{5.2^k})$. Therefore $J(FD_{5.2^{k-1}}) \cong J(FD_{5.2^k})/\omega(H)$ and $\dim_F(J(FD_{5.2^k})) = 5.2^k - 9 + 5.2^k = 5.2^{k+1} - 9$. □

Theorem 4.2. *Let F be a finite field of characteristic 2 with 2^n elements. Let $V_1 = 1 + J(FD_{5.2^k})$ and $V_2 = 1 + \omega(H)$, where $H = Z(D_{5.2^k}) = \{1, r^{5.2^{k-1}}\}$. Then*

1. $$U(FD_{5.2^k})/V_1 \cong \begin{cases} GL(2,F) \times GL(2,F) \times C_{2^n-1}, & \text{if } n \text{ is odd;} \\ GL(2,F_2) \times C_{2^n-1}, & \text{if } n \text{ is even.} \end{cases}$$

2. V_1/V_2 *is a group of order* $2^{(5.2^k-9)n}$.

Proof. Let $H = \{1, r^{5.2^{k-1}}\}$. Then H is a normal subgroup of $D_{5.2^k}$ and $D_{5.2^k}/H \cong D_{5.2^{k-1}}$. Thus $FD_{5.2^k}/\omega(H) \cong FD_{5.2^{k-1}}$ and $\dim_F(\omega(H)) = 5.2^k$. Since $\omega(H)$ is nilpotent, $\omega(H) \subseteq J(FD_{5.2^k})$. Now, $J(FD_{5.2^{k-1}}) \cong J(FD_{5.2^k})/\omega(H)$. By Lemma 4.2, $\dim_F(J(FD_{5.2^k})) = 5.2^{k+1} - 9$. Hence $\dim_F(FD_{5.2^k}/J(FD_{5.2^k})) = 9$.

Now, the 2-regular elements in $D_{5.2^k}$ are 1, r^{2^k}, r^{-2^k}, $r^{2^{k+1}}$ and $r^{-2^{k+1}}$. Hence $m = 5$. Let a be the number of simple components in the Wedderburn decomposition of $FD_{5.2^k}$.

1. If $n = 0 \bmod 4$, then $q \equiv 1 \bmod 5$.

   $T = \{1\} \bmod 5$ and $\{1\}, \{r^{\pm 2^k}\}$, $\{r^{\pm 2^{k+1}}\}$ are the 2-regular F-conjugacy classes. Hence $a = 3$.

2. If $n = 2 \bmod 4$, then $q \equiv -1 \bmod 5$.

   $T = \{1, 4\} \bmod 5$ and $\{1\}, \{r^{\pm 2^k}\}$, $\{r^{\pm 2^{k+1}}\}$ are the 2-regular F-conjugacy classes. Hence $a = 3$.

3. If $n = 1 \bmod 2$, then $q \equiv \pm 2 \bmod 5$.

   $T = \{1, 2, 3, 4\} \bmod 5$ and $\{1\}, \{r^{\pm 2^k}, r^{\pm 2^{k+1}}\}$ are the 2-regular F-conjugacy classes. Thus $a = 2$.

Hence,

$$FD_{5.2^k}/J(FD_{5.2^k}) \cong \begin{cases} F \oplus M(2, F)^2, & \text{if } q \equiv \pm 1 \bmod 5; \\ F \oplus M(2, F_2), & \text{if } q \equiv \pm 2 \bmod 5. \end{cases}$$

Since $V_1 = 1 + J(FD_{5.2^k})$ and $V_2 = 1 + \omega(H)$, $|V_1| = |J(FD_{5.2^k})| = 2^{(5.2^{k+1}-9)n}$ and $|V_2| = |\omega(H)| = 2^{(5.2^k)n}$. Hence $|V_1/V_2| = 2^{(5\cdot 2^k - 9)n}$. □

5. Conclusion

For a finite field F, we have given the structures of $U(FD_4)$ and $U(FD_{10})$ in Theorems 2.1 and 3.1 , whereas the structure of $U(FD_8)$, $U(FD_{16})$ and $U(FD_{20})$ are described in Theorems 2.2, 2.3 and 3.2, when F has odd characteristic. The unit groups $U(FD_{2^k})$ and $U(FD_{5.2^k})$, when characteristic of F is 2, have been studied in Theorems 4.1 and 4.2.

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